Speculations on some characteristic properties of numbers*

Leonhard Euler

§1. There is no doubt that the multitude of all the different fractions, which can be constituted between the terms 0 and 1, is infinite; whence, because the multitude of all the numbers together is also infinite, it is apparent for the multitude of all fractions to be still infinitely greater than this; for, between two numbers, differing by unity, innumerable different fractions may be admitted. Here it is taken that the denominators of the fractions can be increased up to infinity: and if a term is picked which the numerators are not allowed to exceed, then certainly the number of fractions, which can be constituted between the terms 0 and 1, will be determinate. But, as there is some limit, however large this number will be which is taken for the denominators, at first sight this question does not seem that difficult; truly though if we carefully consider the matter, so many difficulties occur that a perfect solution of this question seems hardly possible to hope for here.

§2. Now because the fractions, which we are inquiring into here, must all be different from each other, from any particular denominator no other fractions may be formed unless not only the numerators of them are less than the denominator, but also they are prime to it, as otherwise they could be reduced to a simpler form. Thus as the fraction $\frac{15}{24}$ may be reduced to $\frac{5}{8}$, this fraction cannot be counted with the denominator $= 24$, since it has already been counted with the denominator 8. The whole matter is thus reduced to, for any particular denominator, which may $= D$, assigning the multitude of numbers less than it and which have no common divisor with it, and of course these can be taken as the numerators just for one particular denominator. Thus for the denominator 24 no other numbers are admitted as numerators besides $1, 5, 7, 11, 13, 17, 19, 23$, the multitude of which is 8, and the ratio of this depends on the composition of the number 24. For if the denominator D were a prime number, then certainly all the numbers less than it, the multitude of which is $D - 1$, serve as suitable numerators. Namely the more divisors the denominator D has, the more greatly

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is restricted the multitude of numerators.

§3. The question arises here: for any given number D, to assign the multitude of numbers which are less than it and also prime to it. So that this can be presented more easily, let the character πD denote that multitude of numbers which are less than D and which have no common divisor with it. And indeed it is clear first that if D were a prime number, then $\pi = D - 1$. As we have previously examined the composite numbers, we shall tabulate the values of this character πD for all numbers not greater than one hundred:

$\pi 1 = 0$	$\pi 21 = 12$	$\pi 41 = 40$	$\pi 61 = 60$	$\pi 81 = 54$
$\pi 2 = 1$	$\pi 22 = 10$	$\pi 42 = 12$	$\pi 62 = 30$	$\pi 82 = 40$
$\pi 3 = 2$	$\pi 23 = 22$	$\pi 43 = 42$	$\pi 63 = 36$	$\pi 83 = 82$
$\pi 4 = 2$	$\pi 24 = 8$	$\pi 44 = 20$	$\pi 64 = 32$	$\pi 84 = 24$
$\pi 5 = 4$	$\pi 25 = 20$	$\pi 45 = 24$	$\pi 65 = 48$	$\pi 85 = 64$
$\pi 6 = 2$	$\pi 26 = 12$	$\pi 46 = 22$	$\pi 66 = 20$	$\pi 86 = 42$
$\pi 7 = 6$	$\pi 27 = 18$	$\pi 47 = 46$	$\pi 67 = 66$	$\pi 87 = 56$
$\pi 8 = 4$	$\pi 28 = 12$	$\pi 48 = 16$	$\pi 68 = 32$	$\pi 88 = 40$
$\pi 9 = 6$	$\pi 29 = 28$	$\pi 49 = 42$	$\pi 69 = 44$	$\pi 89 = 88$
$\pi 10 = 4$	$\pi 30 = 8$	$\pi 50 = 20$	$\pi 70 = 24$	$\pi 90 = 24$
$\pi 11 = 10$	$\pi 31 = 30$	$\pi 51 = 32$	$\pi 71 = 70$	$\pi 91 = 72$
$\pi 12 = 4$	$\pi 32 = 16$	$\pi 52 = 24$	$\pi 72 = 24$	$\pi 92 = 44$
$\pi 13 = 12$	$\pi 33 = 20$	$\pi 53 = 52$	$\pi 73 = 72$	$\pi 93 = 60$
$\pi 14 = 6$	$\pi 34 = 16$	$\pi 54 = 18$	$\pi 74 = 36$	$\pi 94 = 46$
$\pi 15 = 8$	$\pi 35 = 24$	$\pi 55 = 40$	$\pi 75 = 40$	$\pi 95 = 72$
$\pi 16 = 8$	$\pi 36 = 12$	$\pi 56 = 24$	$\pi 76 = 36$	$\pi 96 = 32$
$\pi 17 = 16$	$\pi 37 = 36$	$\pi 57 = 36$	$\pi 77 = 60$	$\pi 97 = 96$
$\pi 18 = 6$	$\pi 38 = 18$	$\pi 58 = 28$	$\pi 78 = 24$	$\pi 98 = 42$
$\pi 19 = 18$	$\pi 39 = 24$	$\pi 59 = 58$	$\pi 79 = 78$	$\pi 99 = 60$
$\pi 20 = 8$	$\pi 40 = 16$	$\pi 60 = 16$	$\pi 80 = 32$	$\pi 100 = 40$

§4. From this table it is clear that the denominator 2 provides only one fraction between 0 and 1, namely $\frac{1}{2}$; the denominator 3 indeed gives 2; and 4 gives the two fractions $\frac{1}{3}$ and $\frac{3}{4}$, and so on. Whence if we want to continue the denominators not beyond 10, the number of all these fractions will be 31; if rather we should continue to 20, the number is 127; and proceeding to 30 the sum of the fractions gives 277, as the following table indicates.

Max. denom.	Num. fract.
10	31
20	127
30	277
40	489
50	773
60	1101
70	1493
80	1975
90	2489
100	3043

§5. But then clearly if we wanted to admit all fractions for the denominator $= 10$ the maximum number of all the fractions would be $1+2+3+4+\cdots+9 = 45$. Then those which admit reduction should be excluded. Therefore first the fractions $\frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}$, which are of course $= \frac{1}{2}$, will be excluded; then indeed $\frac{2}{6}$ and $\frac{3}{9}$, of course $= \frac{1}{3}$; and likewise $\frac{4}{6}$ and $\frac{6}{9}$ as they can be made $= \frac{2}{3}$; also $\frac{2}{8}$ and $\frac{6}{8}$; and finally $\frac{2}{10}, \frac{4}{10}, \frac{6}{10}, \frac{8}{10}$, and the number of all these is 14, and when this is subtracted from 45, 31 remains. Though for most denominators which we will want to admit, this enumeration would be too extended, nevertheless, let us see how it can be carried out.

§6. Thus were D the maximum denominator which we admit, the number of all fractions will plainly be

$$= 1+2+3+4+\cdots+(D-1) = \frac{DD-D}{2}.$$

Then all the fractions should be excluded whose value is $\frac{1}{2}$, aside from $\frac{1}{2}$ itself. To this end, D is divided by 2 and the quotient, either exactly or approximate and less, shall be $= \alpha$, and it is clear that the number of fractions which are to be excluded is $= \alpha - 1$. Then for the fractions $\frac{1}{3}$ and $\frac{2}{3}$, let $\frac{D}{3} = \beta$, with β denoting either exactly or approximate and less, and the number of fractions to be excluded will be $= 2(\beta-1) = (\beta-1)\pi : 3$. In a similar way if we put $\frac{D}{4} = \gamma$; then indeed likewise $\frac{D}{5} = \delta$, $\frac{D}{6} = \epsilon$, etc., namely until the quotients go past unity, the numbers of fractions to be excluded will then be

$$(\gamma-1)\pi : 4, \quad (\delta-1)\pi : 5, \quad (\epsilon-1)\pi : 6, \quad \text{etc.}$$

With these removed, the multitude of fractions which are being searched for which remain will be:

$$\frac{DD-D}{2} - (\alpha-1)\pi 2 - (\beta-1)\pi 3 - (\gamma-1)\pi 4 - (\delta-1)\pi 5 - \text{etc.}$$

So if it were $D = 20$, we will have

$$\begin{aligned}
&\frac{20}{2} = \alpha = 10, \quad \frac{20}{3} = \beta = 6, \quad \frac{20}{4} = \gamma = 5, \quad \frac{20}{5} = \delta = 4, \\
&\frac{20}{6} = \epsilon = 3, \quad \frac{20}{7} = \zeta = 2, \quad \frac{20}{8} = \eta = 2, \\
&\frac{20}{9} = \theta = 2, \quad \frac{20}{10} = \iota = 2.
\end{aligned}$$

Thus here, because $\frac{DD-D}{2} = 190$, the multitude of different fractions will be

$$\begin{aligned}
&190 - 9 \cdot 1 - 5 \cdot 2 - 4 \cdot 2 - 3 \cdot 4 - 2 \cdot 2 - 1 \cdot 6 - 1 \cdot 4 - 1 \cdot 6 - 1 \cdot 4 \\
&= 190 - 63 = 127,
\end{aligned}$$

as we have found above.

§7. Therefore all of this investigation rests on this point, that, for any given number D, the value of the character πD needs to be found. And indeed we should first note, as before, that if D is a prime number then it will be $\pi D = D - 1$. Truly if D is a composite number, the determination of the character πD does not turn out to be too arduous; namely it will depend on the factors from which the number D is comprised.

§8. Thus let π denote any prime number, so that it would be $\pi p = p - 1$, and let us search for the value of πp^2; it is certainly clear at once that not all the numbers less than it, the multitude of which is $pp - 1$, are prime to it, but just those numbers should be excluded which are divisible by p, which are: $p, 2p, 3p, 4p, \text{etc.}, (p-1)p$. But the multitude of these is $p - 1$, and when this number is subtracted from $pp - 1$, $p(p - 1)$ remains, so that it would be $\pi pp = (p - 1)p$. In a similar way, if it were $D = p^3$, the multitude of numbers less than it is $p^3 - 1$, whence those should be excluded which are divisible by p, which are

$$p, 2p, 3p, 4p, \text{etc.}, p(pp - 1),$$

the multitude of which is $pp - 1$, hence it will be

$$\pi p^3 = p^3 - 1 - (pp - 1) = p^3 - pp = (p - 1)pp.$$

From this it is now easy to see the for any power, it will in general be $\pi p^n = (p - 1)p^{n-1}$.

§9. Now let q be another prime number different than p, and let us look for the value of πpq. First of all therefore, the multitude of numbers less than pq is $pq - 1$, and thus all those should be excluded from this which are divisible by either p or by q. Indeed the multiples of p will be

$$p, 2p, 3p, 4p, \ldots, p(q - 1),$$

the multitude of which is $q - 1$. In the same way, the multitude of the multiples of q will be $p - 1$, and since these would all be different from the first, the

multitude of all numbers to be excluded will be $p+q-2$, so that it follows here that

$$\pi pq = pq - 1 - (p+q-2) = pq - p - q - 1 = (p-1)(q-1);$$

from which we obtain this excellent Theorem: If p and q are different prime numbers, it will always be

$$\pi : pq = (p-1)(q-1).$$

This can be further extended in the same way, that if as well r and s were prime numbers different from the first, it will be

$$\pi pqr = (p-1)(q-1)(r-1) \text{ and}$$
$$\pi pqrs = (p-1)(q-1)(r-1)(s-1).$$

§10. Let us now investigate the value of this formula: πppq, where the multitude of all numbers less than ppq is $ppq-1$, from which first all the multiples of p should be excluded, the multitude of which is $pq-1$; then indeed the multitude of numbers divisible by q is $pp-1$, between which however the numbers occur

$$pq, 2pq, 3pq, \text{etc.}, pq(p-1)$$

which are also divisible by p. Because we want to exclude them here, this should be removed from the final count, so that this many will remain $pp-1-(p-1) = pp-p$, whence we will then obtain

$$\pi ppq = ppq - 1 - (pq-1) - (pp-p) = (p-1)(q-1)p.$$

Like how it is

$$p(p-1) = \pi pp \quad \text{and} \quad q-1 = \pi : q,$$

this theorem can here be obtained: If p and q are different prime numbers, then it will be

$$pippq = \pi pp \cdot \pi q = p(p-1)(q-1).$$

§11. In a similar way it is hardly difficult to see that

$$\pi p^n q = \pi p^n \pi q = p^{n-1}(p-1)(q-1).$$

For, because the multitude of numbers less than it is $p^n q - 1$, first here all the multiples of q should be excluded, the number of which is $p^n - 1$, and the multitude that will remain is $p^n q - p^n$. Besides indeed we should also exclude all the numbers divisible by p, the multitude of which is $p^{n-1}q - 1$, and $p^n q - p^n - p^{n-1}q + 1$ would remain. To this, however, all the terms divisible by pq should be added, the multitude of which is $p^{n-1} - 1$, from which one gathers

$$\begin{aligned} \pi p^n q &= p^n q - p^n - p^{n-1}q + p^{n-1} \\ &= p^{n-1}(pq - p - q + 1) = p^{n-1}(p-1)(q-1). \end{aligned}$$

§12. In a not at all dissimilar way, if a number D were a product from two powers of any different prime numbers p and q, so that it would thus be $D = p^\alpha q^\beta$, then it will be

$$\pi p^\alpha q^\beta = p^{\alpha-1} q^\beta (p-1)(q-1);$$

and then generally, if the letters p, q, r, s denote prime numbers different from each other, it will be

$$\pi p^\alpha q^\beta r^\gamma s^\delta = p^{\alpha-1} q^{\beta-1} r^{\gamma-1} s^{\delta-1} (p-1)(q-1)(r-1)(s-1)$$

from which one realizes that it will also be

$$\pi p^\alpha q^\beta r^\gamma s^\delta = \pi p^\alpha \cdot \pi q^\beta \cdot \pi r^\gamma \cdot \pi s^\delta.$$

Because of this, if only the values of the character πD were found for all powers of prime numbers, then it is perfectly clear that from these, the values of the character π of all numbers could be readily assigned.

§13. If, by means of these Theorems, one wants to investigate the values for arbitrarily large numbers, the goal will be obtained most quickly if one resolves the given number D into factors which are prime to each other, either prime numbers or not. For in fact if it were $D = PQRS$ etc. and these factors P, Q, R, S have no common divisors, then it will always be

$$\pi PQRS = \pi P \cdot \pi Q \cdot \pi R \cdot \pi S.$$

Namely if it were $D = PQRS$ etc., and these factors P, Q, R, S have no common divisors, then it will always be

$$\pi PQRS = \pi Q \cdot \pi Q \cdot \pi R \cdot \pi S.$$

Like if this number were proposed: $D = 360$, because $360 = 9 \cdot 40$, it will be $\pi 360 = \pi 9 \cdot \pi 40 = 6 \cdot 16 = 96$.

§14. But if indeed the progression of these numbers, which were exhibited in the table given above, are considered, which is $0, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4$, etc., one can find no clear order in the terms of it; yet in the progression of numbers each term of which exhibits the sums of the divisors of the natural numbers, I did succeed in detecting a characteristic order. Thus at least, if from these numbers such a series were formed:

$$1x^2 + 2x^3 + 2x^4 + 4x^5 + 2x^{10} + \text{etc.}$$

the general term of which is signified by our method as $x^n \pi n$, one sees that the character of it, or even the sum, might be expressed in in some way by known quantities, either algebraic or transcendental. Therefore it is worth the greatest effort to inquire into the nature of this progression, since here the science of numbers can be enriched with a not negligible increase.

§15. However from the general form given in §12, a much easier rule can be deduced, by means of which for any given number N the value of the character πN can be assigned, which we shall explain in the following Problem.

Problem.

Given any number N to find the multitude of all numbers less than it and prime to it.

Solution.

§16. For any number N, it can always be represented in such a form as $N = p^\alpha q^\beta r^\gamma s^\delta$ etc., with p, q, r, s being prime numbers. We have also found for it then to be

$$\pi N = p^{\alpha-1} q^{\beta-1} r^{\gamma-1} s^{\delta-1} (p-1)(q-1)(r-1)(s-1).$$

Then it will therefore be

$$\frac{\pi N}{N} = \frac{(p-1)(q-1)(r-1)(s-1)}{pqrs},$$

from which it follows that

$$\pi N = \frac{N(p-1)(q-1)(r-1)(s-1)}{pqrs};$$

so that there does not have to be any more work to know the exponents α, β, γ, but rather it suffices to just investigate all the different prime numbers p, q, r, s by which the given number N is divisible; with these known, the multitude of numbers which are less than N and also prime to it will be

$$\pi N = \frac{N(p-1)(q-1)(r-1)(s-1)}{pqrs}.$$

§17. So if, e.g., this number were proposed: $N = 9450$, the prime numbers which divide this number are $2, 3, 5, 7$; since it does not admit division by any other, it will therefore be

$$\pi 9450 = \frac{9450 \cdot 1 \cdot 2 \cdot 4 \cdot 6}{2 \cdot 3 \cdot 5 \cdot 7} = 2160.$$

§18. Thus if ever N has just a single prime divisor p, which happens when either when N is equal to p itself, or some power of it; then it will therefore always be $\pi N = \frac{N(p-1)}{p}$. Namely if it were $N = p$, it will be $\pi N = p - 1$; and if it were $N = p^n$, then it will be $\pi N = p^{n-1}(p-1)$, as we have found above. But if however N admits two prime divisors p and q, then it will be $\pi N = \frac{N(p-1)(q-1)}{pq}$. Thus if N has no other divisors besides 2 and 3, it will be $\pi N = \frac{N}{3}$. Such numbers up to one hundred are

$$6, 12, 18, 24, 36, 48, 54, 72, 96.$$

§19. For let us take the number N to have the prime divisors p, q and r, different from each other, and besides these no others; and because the multitude of all numbers not greater than it is $= N$, and therefore some number will be divisible by p, q and r, where first all shall be excluded which are divisible by p, the multitude of which would be $\frac{N}{p}$, and with these deleted the multitude of the remaining will be $N - \frac{N}{p} = \frac{N(p-1)}{p}$; from this now we should exclude all which are divisible by q, the multitude of which would be $\frac{N(p-1)}{pq}$, and now there will remain $\frac{N(p-1)(q-1)}{pq}$. Finally now those which are divisible by r should be excluded, the multitude of which would be $\frac{1}{r}$ part of this number. With these deleted, the number of the remaining will be $\frac{N(p-1)(q-1)(r-1)}{pqr}$; and in this way our rule has been firmly demonstrated.

§20. But nevertheless, this rule provides no help to us on the nature of the progression which the numbers πN constitute, and which is:

1	2	3	4	5	6	7	8	9	10	11	12
0	1	2	2	4	2	6	4	6	4	10	4

which is be explored. Certainly, if we adjoin powers of the indefinite quantity x, and we set

$$s = 1x^2 + 2x^3 + 2x^4 + 4x^5 + 2x^6 + 6x^7 + 4x^8 + 6x^9 + 4x^{10} + \text{etc.}$$

from it we can form the following series:

$$\frac{\int sds}{x} = \frac{xx}{2} + \frac{2x^3}{3} + \frac{x^4}{2} + \frac{4x^5}{5} + \frac{x^6}{3} + \frac{6x^7}{7} + \frac{x^8}{2} + \frac{2x^9}{3} + \frac{2x^{10}}{5} + \text{etc.}$$

where all the coefficients are contained in the formula $\frac{(p-1)(q-1)(r-1)}{pqr}$.

§21. Now, all those powers of x will have the same coefficient $\frac{1}{2}$ whose exponents admit just one prime divisor, and thus are powers of two, namely

$$x^2, x^4, x^8, x^{16}, x^{32}, x^{64}, \text{etc.}$$

Then all the powers whose exponents are ranks of three, which are x^3, x^9, x^{27}, would all have the same coefficient $\frac{2}{3}$. In a similar way $\frac{4}{3}$ will be the common coefficient of the powers x^5, x^{25}, x^{125}, etc. And truly $\frac{1}{3} = \frac{1}{2} \cdot \frac{2}{3}$ will be the common coefficient of all the powers whose exponents involve exactly the two prime numbers 2 and 3, which are $x^6.x^{12}, x^{18}, x^{24}, x^{36}$, etc. And the same kind of thing happens for the other exponents, which involve either pairs, or triples, or quadruples of prime numbers. Moreover when more prime numbers occur in the exponents, the series of powers, which enjoy common coefficients, will be more plentiful.

§22. Thus in this order the simplest series are those whose constitute a geometric progression, of which type is $x + x^2 + x^4 + x^8 + x^{16} + \text{etc.}$, but even

the sum of this series has still not been able to be found in any way, or even to be reduced to some integral formula, and at the very least it is hoped that some certain order can be found in this series in general, from which at least the following terms could be determined from the preceding; this rightly would be seen as all the more remarkable, since the coefficient of any power x^n can nevertheless be assigned easily.